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Analysis and Computation of Extremum Points with Application to Boundary-Layer Stability

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A method for analysis and computation of derivatives and extremum points of variable-coefficients differential eigenvalue problems is presented. The method utilizes the orthogonality of the adjoint eigenfunctions to the homogenous part of the once or more differentiated problem to derive an analytical expression for the rate of change of eigenvalue with respect to a free parameter. The extremum point can be analyzed and computed by setting and driving, respectively, the first rate of change of the eigenvalue with respect to the free parameters to zero. Higher order derivatives can be computed by solving, sequentially, sets of inhomogeneous two-point boundary value problems. The method is applied to analyze and compute the most amplified inviscid instability wave in two-dimensional compressible boundary layers and the most amplified viscous instability wave in three-dimensional incompressible boundary layers. It is shown analytically that while the most-amplified spatial instability wave in two-dimensional incompressible boundary layer is two dimensional, the corresponding most amplified wave in three-dimensional boundary layer is generally oblique. It is also shown analytically that the most-amplified disturbance in three-dimensional boundary layer is generally a traveling disturbance. Furthermore, it is shown analytically that the inviscid growth rate is an extremum point. © 1996 Academic Press, Inc.

1. INTRODUCTION

Ordinary differential eigenvalue problems with variable coefficients arise in modeling various phenomena in science and engineering. In order to understand the nature of such modeled phenomena, it is often desired to analyze and quantify the interrelationships of the various free parameters that constitute the dispersion relation associated with the differential eigenvalue problem. The free parameters in a dispersion relation include the eigenvalue as well as other free parameters such as material or media properties, geometric dimensions or properties, wave characteristics, etc. The extremum values of the eigenvalue over all values of one or more free parameters of the dispersion relation of a differential eigenvalue problem have special significance in applications. In addition, the rates of change of eigenvalue with respect to one or more free parameters of the dispersion relation of a differential eigenvalue problem are significant quantities in optimization procedures.

In this work, we present a method for analyzing and computing the rates of change of eigenvalues with respect to differential-eigenvalue-problem parameters as well as computing the extremum points of differential eigenvalue problems. The method is applied to problems from hydrodynamic stability of boundary layers. Efficient computation of the rates of change of some quantity with respect to parameters is essential in sensitivity analysis and optimization studies. While the rates of change of a quantity with respect to a parameter can be computed using two-point finite differences, this method has a low order of accuracy. Moreover, higher-order-finite-difference approximations are costly for practical problems both in terms of computer and user's time.

In hydrodynamic stability, it is known that the most amplified temporal instability wave in two-dimensional incompressible boundary layers is two dimensional. On the other hand, the most amplified waves in three-dimensional boundary layers are oblique. Furthermore, the extensive numerical calculations of Mack [1] revealed that the most amplified first-mode waves in supersonic two-dimensional boundary layers are oblique, whereas the most amplified second-mode waves in supersonic two-dimensional boundary layers are two dimensional. However, most of these findings have no theoretical basis to support them. In this work, we present a tool for such a theoretical basis by deriving an analytical condition which characterizes the extremum growth rate of instability waves. The theoretical basis for some of the above discussed findings is also presented.

In stability studies the maximum disturbance growth rate over all spanwise wave numbers or frequencies or both must be considered in order to determine whether a certain effect is stabilizing or destabilizing [1, 2]. The maximum growth rate over all spanwise wavenumbers is also needed in transition–prediction codes to implement the envelope method of calculating the amplification factor. In this work, we also present a method for efficient and robust computation of the maximum growth rate of instability waves over all values of one or more parameters. Although the method

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is applied to hydrodynamic stability, it can be used, in its general form, to efficiently analyze and compute the derivatives and extremum points of differential eigenvalue problems.

2. GENERAL ANALYSIS

We consider a system of first-order homogenous ordinary differential equations expressed as

$$D\zeta = G(x; P, c_i)\zeta, \quad i = 1, 2, ..., \tag{1}$$

where $D = d/dy$, P is the eigenvalue, and c_i are other free parameters. Higher powers of P and c_i are possible. If we have n equations, then ζ is a column vector with n components and G is a square matrix of order $n \times n$ with variable coefficients. The general homogeneous boundary conditions are

$$A(P, c_i)\zeta(a) + B(P, c_i)\zeta(b) = 0, \tag{2}$$

where $y = a$ and $y = b$ are the two boundaries of the problem domain. Without loss of generality, assume that $a < b$. The constant coefficient matrices A and B are of order $n \times n$ with the sum of their ranks equal to n. Assume that the unknown eigenvalue is P. Our interest is in the rates of change of P with respect to c_i, i.e., $\partial P/\partial c_i$, $\partial^2 P/\partial c_i \partial c_j$, etc. We are particularly interested in the values of P and c_i at which P is an extremum with respect to c_i, i.e., the values at which

$$\frac{\partial P}{\partial c_i} = 0. \tag{3}$$

The usual method of computing the extremum value of P with respect to c_1, for example, consists of calculating numerically the rate of change of P with respect to c_1, i.e., $\partial P/\partial c_1$. This calculation is usually performed using finite differences. Then iteration is performed to drive $\partial P/\partial c_1$ to zero at which point P is an extremum point. The iteration requires the second rate of change of P with respect to c_1, i.e., $\partial^2 P/\partial c_1^2$ which is also calculated numerically using finite differences. This method has a low order of accuracy and it is time consuming when compared to the method that we present in this paper. Masad and Malik [3] presented a method for computing the extremum points of variable-coefficients differential eigenvalue problems and applied it to compute the maximum growth rate of hydrodynamic instability waves. The method of Masad and Malik [3] is a "one-shot" method which augments the original and differentiated systems by trivial equations for the rates of change of free parameters and solves the resulting nonlinear system, subject to the original and differentiated boundary conditions as well as the associated normalization conditions. The method presented in this paper is different than the method of Masad and Malik [3].

The present method of analyzing and computing the derivatives and extremum values of P over all values of c_i starts by multiplying Eq. (1) by ζ^{*T} and integrating the result over the domain of the problem from $y = a$ to $y = b$. The result is

$$\int_a^b \zeta^{*T}(D\zeta - G\zeta)\, dy = 0. \tag{4}$$

Integrating by parts and rearranging, we obtain

$$\zeta^{*T}\zeta\Big|_a^b - \int_a^b (D\zeta^{*T} + \zeta^{*T}G)\zeta\, dy = 0.$$

The adjoint equations [4, 5] are defined by setting

$$D\zeta^{*T} + \zeta^{*T}G = 0,$$

or after transposing,

$$D\zeta^* = -G^T\zeta^*. \tag{5}$$

Then, it follows that

$$\zeta^{*T}\zeta\Big|_a^b = 0. \tag{6}$$

The boundary conditions for the adjoint problem are formulated by using Eq. (2) in Eq. (6) and requiring the resulting terms to vanish independently. Next, we differentiate Eqs. (1) and (2) with respect to c_i. The results is

$$D\left(\frac{\partial\zeta}{\partial c_i}\right) = G\frac{\partial\zeta}{\partial c_i} + \frac{\partial G}{\partial c_i}\zeta \tag{7}$$

$$A\frac{\partial\zeta}{\partial c_i}(a) + B\frac{\partial\zeta}{\partial c_i}(b) + \frac{\partial A}{\partial c_i}\zeta(a) + \frac{\partial B}{\partial c_i}\zeta(b) = 0. \tag{8}$$

By specifying c_i and solving the eigenvalue problem governed by Eqs. (1) and (2), we obtain P and ζ. If we substitute P and ζ into Eqs. (7) and (8), we obtain an inhomogeneous problem. However, the corresponding homogeneous problem has the same form as Eqs. (1) and (2), except that ζ in Eqs. (1) and (2) is replaced by $\partial\zeta/\partial c_i$ in the homogeneous parts of Eqs. (7) and (8). Because Eqs. (1) and (2) form an eigenvalue problem (ζ is unique within an arbitrary multiplicative constant), then ζ is a solution for the homogeneous parts of Eqs. (7) and (8). In addition, ζ^* is also an adjoint to the homogeneous parts of Eqs. (7) and (8). Therefore (see Eq. (4)),

$$\int_a^b \zeta^{*T}\left[D\left(\frac{\partial \zeta}{\partial c_i}\right) - G\frac{\partial \zeta}{\partial c_i}\right] dy = 0. \tag{9}$$

Next, we multiply the inhomogeneous equation (7) by ζ^{*T} and integrate over the domain of the problem from $y = a$ to $y = b$. We use the boundary conditions (2) and those of the adjoint problem and then use the identity (9) to obtain

$$\zeta^{*T}(b)\frac{\partial \zeta}{\partial c_i}(b) - \zeta^{*T}(a)\frac{\partial \zeta}{\partial c_i}(a) - \int_a^b \zeta^{*T}\frac{\partial G}{\partial c_i}\zeta\, dy = 0. \tag{10}$$

Equations (10) are the solvability conditions for the inhomogeneous system that is governed by Eqs. (7) and (8); Eqs. (10) can be solved for $\partial P/\partial c_i$ as

$$\frac{\partial P}{\partial c_i} = \tau_i, \tag{11}$$

where τ_i are the corresponding constants which depend, in general, on P, c_i, ζ, and ζ^*. Equations (11) can be used to calculate $\partial P/\partial c_i$. Furthermore, the extremum value of P over all values of c_i is characterized by $\partial P/\partial c_i = 0$. To calculate this extremum value we assume initial guesses for c_i, then Eqs. (1) and (2) are solved to obtain P and ζ. Then P is used in the adjoint problem (which has the same eigenvalue P) and ζ^* is calculated. Next, τ_i (the right-hand side of (11)) are evaluated. If τ_i are different from zero (which means that $\partial P/\partial c_i$ are different from zero) then the values of c_i are updated according to the errors τ_i, Eqs. (1) and (2) are resolved, the adjoint problem is also resolved, τ_i are reevaluated, and so on until τ_i are driven to zero within the prescribed tolerances. When τ_i are zero, so are $\partial P/\partial c_i$, and c_i have values that result in P being a maximum or a minimum.

In the iteration scheme which aims at driving τ_i (i.e., $\partial P/\partial c_i$) to zero and, instead of calculating the first derivative of τ_i with respect to c_j (i.e., $\partial \tau_i/\partial c_j$ or $\partial^2 P/\partial c_i \partial c_j$) using finite differences, we differentiate Eqs. (7) and (8) with respect to c_j, the result is

$$D\left(\frac{\partial^2 \zeta}{\partial c_i \partial c_j}\right) = G\frac{\partial^2 \zeta}{\partial c_i \partial c_j} + \frac{\partial G}{\partial c_j}\frac{\partial \zeta}{\partial c_i} + \frac{\partial G}{\partial c_i}\frac{\partial \zeta}{\partial c_j} + \frac{\partial^2 G}{\partial c_i \partial c_j}\zeta \tag{12}$$

$$A\frac{\partial^2 \zeta}{\partial c_i \partial c_j}(a) + B\frac{\partial^2}{\partial c_i \partial c_j}(b) + \frac{\partial A}{\partial c_i}\frac{\partial \zeta}{\partial c_j}(a) + \frac{\partial B}{\partial c_i}\frac{\partial \zeta}{\partial c_j}(b) + \frac{\partial A}{\partial c_j}\frac{\partial \zeta}{\partial c_i}(a) + \frac{\partial B}{\partial c_j}\frac{\partial \zeta}{\partial c_i}(b) + \frac{\partial^2 A}{\partial c_i \partial c_j}\zeta(a) + \frac{\partial^2 B}{\partial c_i \partial c_j}\zeta(b) = 0. \tag{13}$$

By knowing P, c_i, $\partial P/\partial c_i$, ζ, and ζ^* we can substitute these values into Eqs. (7) and (8) and solve for $\partial \zeta/\partial c_i$. Then substituting P, c_i, $\partial P/\partial c_i$, ζ, and $\partial \zeta/\partial c_i$ into Eqs. (12) and (13), we obtain an inhomogeneous problem. However, the corresponding homogeneous problem has the same form as Eqs. (1) and (2), except that ζ in Eqs. (1) and (2) is replaced by $\partial^2 \zeta/\partial c_i \partial c_j$ in the homogeneous parts of Eqs. (12) and (13). Therefore, as argued earlier, ζ is a solution for the homogeneous parts of Eqs. (12) and (13) and ζ^* is an adjoint to it. The resulting solvability condition is

$$\zeta^{*T}(b)\frac{\partial^2 \zeta}{\partial c_i \partial c_j}(b) - \zeta^{*T}(a)\frac{\partial^2 \zeta}{\partial c_i \partial c_j}(a) - \int_a^b \zeta^{*T}\left(\frac{\partial G}{\partial c_j}\frac{\partial \zeta}{\partial c_i} + \frac{\partial G}{\partial c_i}\frac{\partial \zeta}{\partial c_j} + \frac{\partial^2 G}{\partial c_i \partial c_j}\zeta\right) dy = 0. \tag{14}$$

Equations (14) contain $\partial^2 P/\partial c_i \partial c_j$ that can be solved analytically. Higher order derivatives of P with respect to c_j can be analyzed and calculated by similar extensions.

By having the first and higher derivatives of P with respect to c_i, it also becomes possible to calculate P due to variations in c_i using a Taylor series expansion around a basic set of free parameters c_{i0}. For example, in the presence of one perturbed free parameter c_1 around c_{10} we obtain

$$P(c_1) = P(c_{10}) + \left.\frac{\partial P}{\partial c_1}\right|_{c_1=c_{10}}(c_1 - c_{10}) + \frac{1}{2}\left.\frac{\partial^2 P}{\partial c_1^2}\right|_{c_1=c_{10}}(c_1 - c_{10})^2 + \cdots \tag{15}$$

3. APPLICATIONS

3.1. *Analysis and Computation of the Most-Amplified Inviscid Instability Wave in Compressible Two-Dimensional Boundary Layers*

The quasi-parallel inviscid instability equations of the compressible disturbed flow have various forms [1]. In this analysis, we use the form given by

$$D\zeta_1 = \frac{i\alpha\rho_m DU_m}{\Gamma}\zeta_1 - \left(\frac{\alpha^2+\beta^2}{\Gamma} + \frac{M_\infty^2\Gamma}{\rho_m}\right)\zeta_2 \tag{16}$$

$$D\zeta_2 = -\Gamma\zeta_1, \tag{17}$$

where $D = d/dy$, ρ_m is the meanflow density, U_m is the meanflow streamwise velocity, and M_∞ is the freestream Mach number. We also have

$$v = \zeta_1 e^{i(\alpha x + \beta z - \omega t)} + cc \tag{18a}$$

$$p = \zeta_2 e^{i(\alpha x + \beta z - \omega t)} + cc \tag{18b}$$

$$\Gamma = -i\rho_m(\omega - \alpha U_m). \tag{19}$$

The boundary conditions at the wall are

$$\zeta_1 = 0 \quad \text{at } y = 0 \tag{20}$$

and in the free-stream, the boundedness of the disturbance requires that

$$\zeta_1 - \tau^{1/2}\zeta_2 = 0, \tag{21}$$

where

$$\tau = \frac{\alpha^2 + \beta^2}{\Gamma_e^2} + M_\infty^2 \tag{22}$$

$$\Gamma_e = -i(\omega - \alpha). \tag{23}$$

If we compare Eqs. (16) and (17) with Eq. (1), then we see that

$$G = \begin{bmatrix} \dfrac{i\alpha\rho_m DU_m}{\Gamma} & -\dfrac{\alpha^2+\beta^2}{\Gamma} - \dfrac{M_\infty^2\Gamma}{\rho_m} \\ -\Gamma & 0 \end{bmatrix}. \tag{24}$$

We have $a = 0$ and $b = \infty$. The adjoint problem is governed by

$$D\zeta^* = -G^{\mathrm{T}}\zeta^*, \tag{25}$$

where

$$\zeta^* = |\zeta_1^* \quad \zeta_2^*|^{\mathrm{T}} \tag{26}$$

and the boundary conditions

$$\zeta_2^* = 0 \quad \text{at } y = 0 \tag{27}$$

$$\zeta_1^* + \tau^{-1/2}\zeta_2^* = 0. \tag{28}$$

Applying the solvability condition (10) with $P = \alpha$ and $c_i = \beta$ yields

$$-\tau^{-1/2}\frac{\partial\tau^{1/2}}{\partial\beta}\zeta_2^*(\infty)\zeta_2(\infty) - \int_0^\infty \left(\zeta_1^*\frac{\partial G_{11}}{\partial\beta}\zeta_1 + \zeta_2^*\frac{\partial G_{21}}{\partial\beta}\zeta_1 + \zeta_1^*\frac{\partial G_{12}}{\partial\beta}\zeta_2\right)dy = 0. \tag{29}$$

It follows from Eq. (29) that

$$\frac{\partial\alpha}{\partial\beta} = \frac{\tau_1}{\tau_2}, \tag{30}$$

where

$$\tau_1 = -\frac{1}{\tau}\frac{\beta}{\Gamma_e^2}\zeta_2^*(\infty)\zeta_2(\infty) + 2\beta\int_0^\infty \frac{\zeta_1^*\zeta_2}{\Gamma}\,dy \tag{31a}$$

and

$$\begin{aligned}\tau_2 = &-\frac{i\zeta_2^*(\infty)\zeta_2(\infty)}{\tau\Gamma_e^2}\left(\frac{\alpha^2+\beta^2}{\Gamma_e} + i\alpha\right) \\ &+ \int_0^\infty \left[\frac{\rho_m DU_m}{\Gamma}\left(i + \alpha\frac{\rho_m U_m}{\Gamma}\right)\zeta_1^*\zeta_1 - i\rho_m U_m\zeta_2^*\zeta_1 \right. \\ &\left. + \left\{-\frac{2\alpha}{\Gamma} + \frac{i\rho_m U_m}{\Gamma^2}(\alpha^2+\beta^2) - iM_\infty^2 U_m\right\}\zeta_1^*\zeta_2\right]dy.\end{aligned} \tag{31b}$$

We calculated the values of $\partial\alpha/\partial\beta$ using Eq. (30) and using central finite differencing through

$$\left(\frac{\partial\alpha}{\partial\beta}\right)_i = \frac{\alpha_{i+1} - \alpha_{i=1}}{\beta_{i+1} - \beta_{i-1}} = \frac{\alpha_{i+1} - \alpha_{i-1}}{2\,\Delta\beta}. \tag{32}$$

The step $\Delta\beta$ is taken to be 0.002. The calculations are for second-mode waves at $M_\infty = 5$, a Prandtl number Pr = 0.72, and a freestream static temperature $T_\infty^* = 50°$ K. The two-point boundary value problem governed by Eqs. (16) and (17) and the boundary conditions (20) and (21) and its adjoint problem are solved numerically using a second-order accurate finite-differences scheme with deferred correction [6]. The iteration on the eigenvalue is performed using a Newton–Raphson scheme. Note that the adjoint problem has the same eigenvalue of the original problem and no iteration on the eigenvalue is needed in the adjoint problem. A comparison of $\partial\alpha/\partial\beta$ calculated from Eqs. (30) and (32) is shown in Table I. The values of $\partial\alpha/\partial\beta$ calculated using the present approach compare well with the values calculated using central differencing and the discrepancy is due to the limited order of accuracy of the central differencing calculation of the derivative. While the central differencing calculation requires solving the two-point boundary value problem twice with iteration on the eigenvalue each time, the present approach requires solving the problem only once with iteration on the eigenvalue and solving the adjoint problem once with no iteration on the eigenvalue. The computational requirements of the present method of evaluating the derivative are almost half those of the central finite differencing with the present method being more accurate. In an optimization code, where the derivatives needs to calculated for a large number of times,

TABLE I

Comparison of the Values of $\partial\alpha/\partial\beta$ Calculated Using the Present Approach and the Central Finite Differencing Approach

			Equation (30)		Equation (32)	
β	α_r	α_i	$\partial\alpha_r/\partial\beta$	$\partial\alpha_i/\partial\beta$	$\partial\alpha_r/\partial\beta$	$\partial\alpha_i/\partial\beta$
0.016	0.1874	−0.006229	−0.01028	0.005942	−0.01000	0.005950
0.036	0.1870	−0.006027	−0.02381	0.01464	−0.02250	0.01465
0.056	0.1864	−0.005619	−0.03906	0.02708	−0.04000	0.02712
0.076	0.1854	−0.004881	−0.05792	0.04898	−0.06000	0.04912

the present approach offers considerable savings in central processing unit (CPU) time and user's time.

The quantity $\partial\alpha/\partial\beta$ is

$$\frac{\partial\alpha}{\partial\beta} = \frac{\partial\alpha_r}{\partial\beta} + i\,\frac{\partial\alpha_i}{\partial\beta} = \text{Real}\left(\frac{\tau_1}{\tau_2}\right) + \text{Imag}\left(\frac{\tau_1}{\tau_2}\right). \tag{33}$$

Therefore, to compute the maximum value of $-\alpha_i$ over all values of β for which $\partial\alpha_i/\partial\beta = 0$, we set the condition

$$\text{Imag}\left(\frac{\tau_1}{\tau_2}\right) = 0. \tag{34}$$

The iteration on β can be performed efficiently in the present method by using the expression $\partial^2\alpha/\partial\beta^2$ which follows from applying Eq. (14) with $P = \alpha$ and $c_i = c_j = \beta$. Satisfying condition (34) ensures that α_i is a maximum or a minimum over all values of β. Note from (31a) that $\beta = 0$ results in $\partial\alpha/\partial\beta = 0$. The corresponding $-\alpha_i$ is a maximum for second-mode waves but it is a minimum for first-mode waves. Our predictions of the maximum growth rate using the present approach are in full agreement (within the prescribed tolerance) with the predictions of the method of Masad and Malik [3]. The maximum value of $-\alpha_i$ over all values of β and ω can be computed by implementing simultaneously condition (34) and the condition $\partial\alpha_i/\partial\omega = 0$, $\partial\alpha_i/\partial\omega$ is the imaginary part of the expression for $\partial\alpha/\partial\omega$ which can be derived.

For temporal stability, α and β are real and ω is complex. Application of the solvability condition (10) with $P = \omega$ and $c_i = \beta$ yields

$$\frac{\partial\omega}{\partial\beta} = \frac{\tau_1}{\tau_3}, \tag{35}$$

where

$$\begin{aligned}\tau_3 = {} & \frac{i\zeta_2^*(\infty)\zeta_2(\infty)}{\tau\Gamma_e^3}(\alpha^2+\beta^2) \\ & + \int_0^\infty \left[\frac{-\alpha\rho_m^2 DU_m}{\Gamma^2}\,\zeta_1^*\zeta_1 + i\rho_m\zeta_2^*\zeta_1 \right. \\ & \left. + i\zeta_1^*\left(M_\infty^2 - i\rho_m\frac{\alpha^2+\beta^2}{\Gamma^2}\right)\zeta_2\right] dy.\end{aligned} \tag{36}$$

Note from (31a) and (35) that $\beta = 0$ results in $\partial\omega/\partial\beta = 0$. The corresponding ω_i is a maximum for second-mode waves but it is a minimum for first-mode waves. Equation (35) can be used to calculate the maximum temporal growth rate at which $\partial\omega_i/\partial\beta = 0$. The group velocity components can be calculated from $\partial\omega_r/\partial\beta$ and $\partial\omega_r/\partial\alpha$. The quantity $\partial\omega/\partial\alpha$ can be calculated from

$$\frac{\partial\omega}{\partial\alpha} = \frac{h_1}{h_2} \tag{37}$$

which follows from

$$\begin{aligned}& -\tau^{-1/2}\frac{\partial\tau^{1/2}}{\partial\alpha}\zeta_2^*(\infty)\zeta_2(\infty) \\ & \quad - \int_0^\infty \left(\zeta_1^*\frac{\partial G_{11}}{\partial\alpha}\zeta_1 + \zeta_2^*\frac{\partial G_{21}}{\partial\alpha}\zeta_1 + \zeta_1^*\frac{\partial G_{12}}{\partial\alpha}\zeta_2\right) dy = 0.\end{aligned} \tag{38}$$

The maximum value of ω_i over all values of α can be calculated by implementing the condition $\partial\omega_i/\partial\alpha = 0$.

3.2. *Analysis and Computation of the Most-Amplified Viscous Instability Wave in Three-Dimensional Incompresible Boundary Layers*

We consider the viscous quasi-parallel instability of incompressible three-dimensional boundary layer with the body-curvature effects neglected. This instability is governed by the Orr–Sommerfeld equation,

$$\frac{d^4 v}{dy^4} + f_1 \frac{d^2 v}{dy^2} + f_2 v = 0, \tag{39}$$

where

$$f_1 = -2(\alpha^2 + \beta^2) - iR(\alpha U_m + \beta W_m - \omega) \tag{40a}$$

$$f_2 = (\alpha^2 + \beta^2)^2 + iR(\alpha U_m + \beta W_m - \omega)(\alpha^2 + \beta^2) + iR(\alpha U''_m + \beta W''_m). \tag{40b}$$

U_m and W_m are the chordwise and spanwise mean-flow velocity components. The boundary conditions require the vanishing of v and dv/dy at $y = 0$ and as $y \to \infty$. For spatial instability of flow over an infinite body, ω and β are real and α is complex. If we let

$$\zeta_1 = v, \quad \zeta_2 = v', \quad \zeta_3 = v'', \quad \zeta_4 = v''',$$

then the resulting first-order system has the form (1), where

$$\zeta = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}^{\mathrm{T}} \tag{41}$$

and

$$G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -f_2 & 0 & -f_1 & 0 \end{bmatrix}. \tag{42}$$

Applying the solvability condition (10) with $P = \alpha$ and $c_i = \beta$ results in

$$\int_0^\infty \zeta_4^* \left(\frac{\partial f_2}{\partial \beta} \zeta_1 + \frac{\partial f_1}{\partial \beta} \zeta_3 \right) dy = 0$$

or

$$\frac{\partial \alpha}{\partial \beta} = \frac{\chi_1}{\chi_2}, \tag{43}$$

where

$$\chi_1 = -\int_0^\infty [\zeta_4^* \zeta_1 \{4\beta(\alpha^2 + \beta^2) + 2i\beta R(\alpha U_m + \beta W_m - \omega) + iRW_m(\alpha^2 + \beta^2) + iRW''_m\} + \zeta_4^* \zeta_3 \{-4\beta - iRW_m\}]\, dy \tag{44a}$$

$$\chi_2 = \int_0^\infty [\zeta_4^* \zeta_1 \{4\alpha(\alpha^2 + \beta^2) + 2i\alpha R(\alpha U_m + \beta W_m - \omega) + iRU_m(\alpha^2 + \beta^2) + iRU''_m\} + \zeta_4^* \zeta_3 \{-4\alpha - iRU_m\}]\, dy. \tag{44b}$$

For two-dimensional flow χ_1 reduces to

$$\chi_1 = -\beta \int_0^\infty [\zeta_4^* \zeta_1 \{4(\alpha^2 + \beta^2) + 2iR(\alpha U_m - \omega)\} - 4\zeta_4^* \zeta_3]\, dy. \tag{45}$$

It is clear from Eq. (45) that $\beta = 0$ results in $\partial\alpha/\partial\beta = 0$. Therefore, $\beta = 0$ corresponds to an extremum growth rate in two-dimensional incompressible boundary layers. This extremum growth rate is a maximum and, therefore, the most amplified spatial instability wave in two-dimensional incompressible boundary layer is two-dimensional. Squire's theorem also provides a proof of this result. Note that for three-dimensional boundary layer with $\beta = 0$, χ_1 reduces to

$$\chi_1 = -iR \int_0^\infty [\zeta_4^* \zeta_1 (\alpha^2 W_m + W''_m) - \zeta_4^* \zeta_3 W_m]\, dy \tag{46}$$

which is in general different from zero and, therefore, the most amplified spatial instability wave in three-dimensional incompressible boundary layer is generally oblique. The maximum growth rate is characterized by $\partial\alpha_i/\partial\beta = 0$, or

$$\mathrm{Imag}\left(\frac{\chi_1}{\chi_2}\right) = 0. \tag{47}$$

This condition can be used to determine the value of β which gives the maximum growth rate $-\alpha_i$ over all values of β.

For temporal stability, application of the condition (10) with $P = \omega$ and $c_i = \beta$ results in

$$\frac{\partial \omega}{\partial \beta} = \frac{\chi_1}{\chi_3}, \tag{48}$$

where

$$\chi_3 = \int_0^\infty [-iR(\alpha^2 + \beta^2)\zeta_4^* \zeta_1 + iR\zeta_4^* \zeta_3]\, dy \tag{49}$$

and the same above theoretical results can be obtained by considering condition (48). Condition (48) can be used to calculate the maximum temporal growth rate at which $\partial\omega_i/\partial\beta = 0$. The group velocity components can be calculated from $\partial\omega_r/\partial\beta$ and $\partial\omega_r/\partial\alpha$. The quantity $\partial\omega/\partial\alpha$ can be calculated from

$$\frac{\partial\omega}{\partial\alpha} = \frac{h_3}{h_4} \tag{50}$$

which follows from

$$\int_0^\infty \zeta_4^* \left(\frac{\partial f_2}{\partial\alpha}\zeta_1 + \frac{\partial f_1}{\partial\alpha}\zeta_3\right) dy = 0, \tag{51}$$

which in turn follows from application of condition (10) with $P = \omega$ and $c_i = \alpha$. The maximum value of ω_i over all values of α can be calculated by implementing the condition $\partial\omega_i/\partial\alpha = 0$.

For spatial stability, the solvability condition (10) with $P = \alpha$ and $c_i = \omega$ results in

$$\int_0^\infty \zeta_4^* \left(\frac{\partial f_2}{\partial\omega}\zeta_1 + \frac{\partial f_1}{\partial\omega}\zeta_3\right) = 0 \tag{52}$$

or

$$\frac{\partial\alpha}{\partial\omega} = \frac{\sigma_1}{\sigma_2}, \tag{53}$$

where

$$\sigma_1 = \int_0^\infty iR\zeta_4^*[(\alpha^2+\beta^2)\zeta_1 - \zeta_3]\,dy \tag{54a}$$

$$\begin{aligned}\sigma_2 = \int_0^\infty [&\zeta_4^*\zeta_1\{(\alpha^2+\beta^2)(4\alpha + iRU_m) \\ &+ 2i\alpha R(\alpha U_m + \beta W_m - \omega) + iRU_m''\} \\ &- \zeta_4^*\zeta_3\{4\alpha + iRU_m\}]\,dy.\end{aligned} \tag{54b}$$

It is clear from Eqs. (53) and (54) that for $\omega = 0$, $\partial\alpha/\partial\omega$ is in general different from zero. Therefore, the most amplified disturbance in the three-dimensional boundary layer, which supports both stationary (zero-frequency) and traveling (nonzero frequency) disturbances, is generally a traveling disturbance. The maximum growth rate is characterized by $\partial\alpha_i/\partial\omega = 0$, or

$$\text{Imag}\left(\frac{\sigma_1}{\sigma_2}\right) = 0. \tag{55}$$

This condition can be used to determine the value of ω which gives the maximum growth rate $-\alpha_i$ over all values of ω.

For spatial stability, application of condition (10) with $P = \alpha$ and $c_i = R$ results in

$$\frac{\partial\alpha}{\partial R} = \frac{\Lambda_1}{\Lambda_2}, \tag{56}$$

where

$$\begin{aligned}\Lambda_1 = -\int_0^\infty [&\zeta_4^*\zeta_1\{i(\alpha U_m + \beta W_m - \omega)(\alpha^2+\beta^2) \\ &+ i(\alpha U_m'' + \beta W_m'')\} \\ &- \zeta_4^*\zeta_3 i(\alpha U_m + \beta W_m - \omega)]\,dy\end{aligned} \tag{57a}$$

$$\begin{aligned}\Lambda_2 = \int_0^\infty [&\zeta_4^*\zeta_1\{4\alpha(\alpha^2+\beta^2) + 3iR\alpha^2 U_m + iRU_m''\} \\ &- \zeta_4^*\zeta_3(4\alpha + iRU_m)]\,dy.\end{aligned} \tag{57b}$$

It is clear from condition (56) that as R approaches infinity, $\partial\alpha/\partial R$ approaches zero. This means that the inviscid growth rate is an extremum point. This extremum is a minimum for the incompressible flow under consideration. Condition (56) can be used to calculate the maximum spatial growth rate over all values of R at which $\partial\alpha_i/\partial R = 0$. The same theoretical results can be reached by considering $\partial\omega/\partial R$ within a temporal stability theory. The condition $\partial\omega_i/\partial R = 0$ can be used to calculate the maximum temporal growth rate over all values of R.

It can be shown that for compressible three-dimensional flow the quantity $\partial\alpha/\partial R$ is given by

$$\frac{\partial\alpha}{\partial R} = \frac{b_1 + (b_2/R^2)}{a_1 R + a_2 + (a_3/R)}, \tag{58}$$

where b_1, b_2, a_1, a_2, and a_3 are $O(1)$ and independent of R. Therefore, as R approaches infinity, $\partial\alpha/\partial R$ approaches zero. The extremum point in the inviscid limit switches from being a minimum at low speeds to being a maximum at high speeds. The same results can be reached by considering $\partial\omega/\partial R$ within a temporal stability theory. The quantity $\partial\omega/\partial R$ has the same form as the right-hand side of Eq. (58).

For spatial instability of fully three-dimensional boundary layer α and β are complex, while ω is real. Using the present approach it is possible to derive analytical expression for $\partial\alpha_r/\partial\beta_r$. The expression can be used to implement the envelope method.

3.3. *Laminar Flow Control and Natural Laminar Flow Optimization*

It is possible to use the present method to compute the rate of change of growth rate of instability wave with respect to the suction velocity or surface temperature. For example, for the stability of three-dimensional flow governed by Eq. (39) application of Eq. (10) with $P = \alpha$ and c_i being the surface suction velocity v_w results in

$$\int_0^\infty \zeta_4^* \left(\frac{\partial f_2}{\partial v_w} \zeta_1 + \frac{\partial f_1}{\partial v_w} \zeta_3 \right) dy = 0 \tag{59}$$

or

$$\frac{\partial \alpha}{\partial v_w} = \frac{q_1}{q_2}, \tag{60}$$

where

$$\begin{aligned} q_1 = \int_0^\infty \Bigg\{ & \zeta_4^* \zeta_3 iR \left(\alpha \frac{\partial U_m}{\partial v_w} + \beta \frac{\partial W_m}{\partial v_w} \right) \\ & - \zeta_4^* \zeta_1 \Bigg[iR \left(\frac{\partial U_m''}{\partial v_w} + \frac{\partial W_m''}{\partial v_w} \right) \\ & + iR(\alpha^2 + \beta^2) \left(\alpha \frac{\partial U_m}{\partial v_w} + \beta \frac{\partial W_m}{\partial v_w} \right) \Bigg] \Bigg\} dy \end{aligned} \tag{61a}$$

$$\begin{aligned} q_2 = \int_0^\infty \{ & \zeta_4^* \zeta_1 [4\alpha(\alpha^2 + \beta^2) + 2\alpha i R(\alpha U_m + \beta w_m - \omega) \\ & + iR(\alpha^2 + \beta^2) U_m + iRU_m''] \\ & + \zeta_4^* \zeta_3 [-4\alpha - iRU_m] \} \, dy. \end{aligned} \tag{61b}$$

The second and higher derivatives $\partial^2\alpha/\partial v_w^2$, $\partial^3\alpha/\partial v_w^3$, etc. can be calculated as explained in Section 2. Then, the growth rate $\sigma = -\alpha_i$ at any suction velocity v_w around a basic suction velocity v_{w0} can be calculated from the Taylor series expansion,

$$\begin{aligned} \sigma(v_w) = \sigma(v_{w0}) + & \left. \frac{\partial \sigma}{\partial v_w} \right|_{v_w = v_{w0}} (v_w - v_{w0}) \\ & + \frac{1}{2} \left. \frac{\partial^2 \sigma}{\partial v_w^2} \right|_{v_w = v_{w0}} (v_w - v_{w0})^2 + \cdots . \end{aligned} \tag{62}$$

The present method can also be applied to compute the rates of change of the growth rate with respect to the aerodynamic shape then it can use these sensitivity derivatives to optimize the shape of aerodynamic configurations in order to delay transition and to achieve large regions of natural laminar flow.

Masad and Malik [7] presented a method for computing the instability eigenvalue subject to perturbations in mean flow parameters without resolving the instability eigenvalue problem. The method consists of using a solvability condition to compute a correction to the eigenvalue due to perturbation in the mean flow parameters. The method of Masad and Malik [7] can be used as a block in a suction or cooling optimization scheme. The application of the present method in this paper is different than the method of Masad and Malik [7].

3.4. *Computation of the Eigenvalue around Base Parameters*

It is possible to use the computed rates of change of the spatial eigenvalue α with respect to parameters such as the spanwise wavenumber β, frequency ω, or Reynolds number R to calculate the value of α around some values of β, ω, or R using a Tyalor series expansion. For example, to calculate α around some Reynolds number R_0 we have

$$\alpha(R) = \sum_{j=0}^{\infty} \frac{1}{j!} \left. \frac{\partial^j \alpha}{\partial R^j} \right|_{R=R_0} (R - R_0)^j. \tag{63}$$

The integrated value of α with respect to R is

$$\begin{aligned} \int_{R=R_1}^{R} \alpha \, dR = \sum_{j=0}^{\infty} & \frac{1}{(j+1)!} \left. \frac{\partial^{(j)} \alpha}{\partial R^{(j)}} \right|_{R=R_0} \\ & \times \{(R - R_0)^{j+1} - (R - R_0)^{j+1}\}. \end{aligned} \tag{64}$$

The N factor is

$$N = -2 \, \mathrm{Imag} \left(\int_{R=R_1}^{R} \alpha \, dR \right). \tag{65}$$

Expansions similar to expansion (63) can be used to calculate $\alpha(\beta)$ and $\alpha(\omega)$ around some values of β_0 and ω_0.

Masad [8] presented a method for computing the instability eigenvalue subject to perturbation in instability parameters such as β and ω without resolving the instability eigenvalue problem. The above-presented method is different than the method of Masad.

5. SUMMARY

A method for analyzing and computing the derivatives and extremum points of variable-coefficients differential eigenvalue problems is presented. The method utilizes the orthogonality of the adjoint eigenfunctions to the differentiated problem to formulate a solvability condition. Theoretical results on the most-amplified instability waves in two- and three-dimensional boundary layers are presented. The method can be used for efficient analysis and computation of the maximum growth rate of instability waves, as well as in laminar flow control and natural laminar flow applications. Furthermore, the method might have some application in the calculation of absolute instabilities.

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